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Exact Solutions of Dirac Equation and Particle Creation in $(1+3)$-Dimensional Robertson-Walker Spacetime

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Abstract The exact solutions of the Dirac equation are discussed for a Robertson-Walker spacetime with asymptotically Minkowskian *in* and *out* regions. We obtain the mode solutions which reduce to positive and negative Minkowskian spinors in asymptotically regions. Using the obtained solutions we compute the density of created particles.

Keywords Dirac equation · Robertson-Walker spacetime · Particle creation

1 Introduction

The behavior of relativistic particles obeying the covariant Dirac equation in Robertson-Walker space-time has been discussed in various contexts [1–8]. One of the most important but difficult problems of the quantum field theory in curved spacetime is the problem of finding analytic solutions of Dirac equation on given backgrounds. Barut and Duru [1], who based their formulation of the Dirac equation in the Robertson-Walker (RW) metric on the assumption of spatially "flat" hypersurfaces, investigated the exact solutions of the Dirac equation for three typical models of expanding universe.

The purpose of the present paper is solution of Dirac equation in a model of RW space-time which is manifestly Minkowskian in the remote past and future. In this model of space-time *in* and *out* vacua are well defined because the scale factor reduces to a constant at the asymptotic regions.

2 Dirac Equation and It's Solutions

Consider a spatially flat RW space with the metric

$$ds^2 = dt^2 - a^2(t)dx^i dx_i, \tag{1}$$

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where a is the scale factor of the expanding Universe. In terms of conformal time parameter given by $\eta = \int dt/a(t)$, the line element (1) to be

$$ds^2 = a^2(\eta)(d\eta^2 - dx^i dx_i). \tag{2}$$

The Dirac equation in the line element (2) is

$$\left(i\gamma^\mu \partial_\mu + i\frac{3}{2}\frac{\dot{a}}{a}\gamma^0 - ma\right)\Psi = 0 \tag{3}$$

dot refers to differentiation with respect to conformal time η and γ's are flat-Dirac matrices

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = \gamma^0\alpha^i, \quad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{4}$$

The Dirac equation (3) becomes

$$(i\gamma^\mu \partial_\mu - ma)\Phi = 0, \tag{5}$$

where

$$\Psi = a^{-3/2}\Phi, \tag{6}$$

by substitution

$$\Phi = (i\gamma^\mu \partial_\mu + ma)\phi, \tag{7}$$

into (5) we arrive at

$$(\Box - i\gamma_0 \dot{a} m + m^2 a^2)\phi = 0. \tag{8}$$

Analogous to Minkowskian spacetime we define the positive and negative frequency modes as follows

$$\phi^{(+)} = N u(m,d) f^{(+)}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{9}$$

$$\phi^{(-)} = N v(m,d) f^{(-)}(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{10}$$

where $d = 1, 2$. Here N is normalization constant and $u(m,d)$ and $v(m,d)$ are flat rest spinors satisfying

$$\gamma^0 u(m,d) = u(m,d), \tag{11}$$

$$\gamma^0 v(m,d) = -v(m,d). \tag{12}$$

$f^{(+)}(\eta)$ and $f^{(-)}(\eta)$ satisfying in the following equations

$$(\partial_\eta^2 + k^2 - i\dot{a}m + m^2a^2) f(\eta) = 0, \tag{13}$$

or

$$(\partial_\eta^2 + k^2 + i\dot{a}m + m^2a^2) f(\eta) = 0. \tag{14}$$

Note that $f^{(+)}$ in the asymptotic regions reduces to positive frequency mode and $f^{(-)}$ reduces to negative frequency mode. In the asymptotic regions as $\eta \to \pm\infty$ (13) and (14) reduce to

$$(\partial_\eta^2 + k^2 + m^2 a^2(\pm\infty)) f(\eta) = 0, \tag{15}$$

with the solutions $f^{(\pm)}(\eta) \approx e^{\mp i\omega_{out}\eta}$ in the *out* region and $f^{(\pm)}(\eta) \approx e^{\mp i\omega_{in}\eta}$ in the *in* region, where

$$\omega_{in}^{out} = (k^2 + m^2 a^2(\pm\infty))^{1/2} = (k^2 + M_{in}^{out\,2})^{1/2}. \tag{16}$$

Then the positive and negative spinors take the forms

$$\begin{aligned} U(k,d,\mathbf{x},\eta) &= N(i\partial_\eta - \mathbf{k}\cdot\gamma + ma)u(m,d) f^{(+)}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= D^{(+)} u(m,d) f^{(+)}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}, \end{aligned} \tag{17}$$

$$\begin{aligned} V(k,d,\mathbf{x},\eta) &= N(-i\partial_\eta + \mathbf{k}\cdot\gamma + ma)v(m,d) f^{(-)}(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= D^{(-)} v(m,d) f^{(-)}(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}, \end{aligned} \tag{18}$$

which in the asymptotic regions as $\eta \to \pm\infty$ reduce to

$$U_{in}^{out}(k,d,\mathbf{x},\eta) \longrightarrow \frac{\gamma^\mu k_\mu + M_{in}^{out}}{\sqrt{2M_{in}^{out}(M_{in}^{out} + \omega_{in}^{out})}} u(m,d) e^{-ik\cdot x}, \tag{19}$$

$$V_{in}^{out}(k,d,\mathbf{x},\eta) \longrightarrow \frac{-\gamma^\mu k_\mu + M_{in}^{out}}{\sqrt{2M_{in}^{out}(M_{in}^{out} + \omega_{in}^{out})}} v(m,d) e^{ik\cdot x}. \tag{20}$$

Then the normalization constance N takes the value

$$N_{in}^{out} = \frac{1}{\sqrt{2M_{in}^{out}(M_{in}^{out} + \omega_{in}^{out})}}. \tag{21}$$

In Dirac representation for γ matrices we have

$$D^{(+)} = \begin{pmatrix} i\partial_0 + ma & 0 & -k_3 & -k_1 + ik_2 \\ 0 & i\partial_0 + ma & -k_1 - ik_2 & k_3 \\ k_3 & k_1 - ik_2 & i\partial_0 + ma & 0 \\ k_1 + ik_2 & -k_3 & 0 & i\partial_0 + ma \end{pmatrix}, \tag{22}$$

$$D^{(-)} = \begin{pmatrix} -i\partial_0 + ma & 0 & k_3 & k_1 - ik_2 \\ 0 & -i\partial_0 + ma & k_1 + ik_2 & -k_3 \\ -k_3 & -k_1 + ik_2 & -i\partial_0 + ma & 0 \\ -k_1 - ik_2 & k_3 & 0 & -i\partial_0 + ma \end{pmatrix}. \tag{23}$$

Then

$$U(k,1,\mathbf{x},\eta) = N \begin{pmatrix} i\dot{f}_k^{(+)} + maf_k^{(+)} \\ 0 \\ k_3 f_k^{(+)} \\ (k_1 + ik_2) f_k^{(+)} \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{24}$$

$$U(k,2,\mathbf{x},\eta)=N\begin{pmatrix}0\\ i\dot{f}_k^{(+)}+maf_k^{(+)}\\ (k_1-ik_2)f_k^{(+)}\\ -k_3f_k^{(+)}\end{pmatrix}e^{i\mathbf{k}\cdot\mathbf{x}},\tag{25}$$

$$V(k,1,\mathbf{x},\eta)=N\begin{pmatrix}k_3f_k^{(-)}\\ (k_1+ik_2)f_k^{(-)}\\ -i\dot{f}_k^{(-)}+maf_k^{(-)}\\ 0\end{pmatrix}e^{-i\mathbf{k}\cdot\mathbf{x}},\tag{26}$$

$$V(k,2,\mathbf{x},\eta)=N\begin{pmatrix}(k_1-ik_2)f_k^{(-)}\\ -k_3f_k^{(-)}\\ 0\\ -i\dot{f}_k^{(-)}+maf_k^{(-)}\end{pmatrix}e^{-i\mathbf{k}\cdot\mathbf{x}}.\tag{27}$$

Suppose that

$$a(\eta)=A+B\tanh(\rho\eta),\qquad A,B,\rho=\text{constants},\tag{28}$$

which is Minkowskian in the far past and future, i.e., $a\longrightarrow(A+B)$ in the in region and $a\longrightarrow(A-B)$ at the out region. This spacetime was first discussed by Bernard and Duncan [9]. Note that this spacetime has no big bang, instead the universe starts out as Minkowskian space, then expands smoothly and ends up as another Minkowskian space. In the limit $A\to 0$ and $0<\eta\ll 1/\rho$, $a(t)\propto t^{1/2}$, so universe behaves like radiation-dominated Friedman cosmology with $K=0$. For scale factor (28) the solutions of (13) which in remote past reduce to positive frequency mode is

$$f_{in}^{(+)}(\eta,a,b,c)=\exp\left[-i\omega_+\eta-\left(\frac{i\omega_-}{\rho}\ln[2\cosh\rho\eta]\right)\right]F(a,b,c,z).\tag{29}$$

Another mode which in far past reduces to negative frequency mode is

$$\begin{aligned}f_{in}^{(-)}(\eta,a,b,c)&=\exp\left[-i\omega_-\eta-\left(\frac{i\omega_+}{\rho}\ln[2\cosh\rho\eta]\right)\right]\\&\quad\times F(a-c+1,b-c+1,2-c,z),\end{aligned}\tag{30}$$

where

$$a=\frac{i\omega_-}{\rho}+\frac{imB}{\rho},\qquad b=1+\frac{i\omega_-}{\rho}-\frac{imB}{\rho},\tag{31}$$

$$c=1-\frac{i\omega_{in}}{\rho},\qquad z=\frac{\tanh(\rho\eta)+1}{2},\tag{32}$$

$$\omega_{in}^{out}=\omega_+\pm\omega_-,\quad \omega_\pm=\frac{1}{2}(\omega_{out}\pm\omega_{in}).\tag{33}$$

Similarly, one may find a complete set of modes of the field that behaving as positive and negative frequency modes in far future

$$f_{out}^{(+)}(\eta,a,b,c)=\exp\left[-i\omega_+\eta-\left(\frac{i\omega_-}{\rho}\ln[2\cosh\rho\eta]\right)\right]\\ \times F(a,b,a+b-c+1,1-z). \tag{34}$$

$$f_{out}^{(-)}(\eta,a,b,c)=\exp\left[i\omega_-\eta+\left(\frac{i\omega_+}{\rho}\ln[2\cosh\rho\eta]\right)\right]\\ \times F(c-a,c-b,c-a-b+1,1-z). \tag{35}$$

Using the following property of hypergeometric functions [10]

$$z(1-z)\frac{d}{dz}F(a,b,c,z)=a\left[\frac{c-a-1}{(b-a-1)}-(1-z)\right]F(a,b,c,z)\\ -\frac{a(c-b)}{(b-a-1)}F(a+1,b-1,c,z) \tag{36}$$

we have

$$\begin{aligned}&\left[i\frac{d}{d\eta}+m(A+B\tanh(\rho\eta))\right]f_{in}^{(+)}\\ &=\exp\left[-i\omega_+\eta-\left(\frac{i\omega_-}{\rho}\ln[2\cosh\rho\eta]\right)\right]\\ &\quad\times\left(\omega_++2\omega_-z-\omega_-+(\omega_-+mB)\left(2-2z-\frac{\omega_++mB}{mB}\right)\right.\\ &\quad\left.+m(A-B)+2mBz\right)F(a,b,c,z)\\ &\quad+\frac{(\omega_-+mB)(\omega_+-mB)}{mB}F(a+1,b-1,c,z)\\ &=\exp\left[-i\omega_+\eta-\left(\frac{i\omega_-}{\rho}\ln[2\cosh\rho\eta]\right)\right]\\ &\quad\times\left[\left(mA-\frac{\omega_-\omega_+}{mB}\right)F(a,b,c,z)\right.\\ &\quad\left.+\frac{(\omega_-+mB)(\omega_+-mB)}{mB}F(a+1,b-1,c,z)\right]\end{aligned}$$

It's easy to check that

$$\left(mA-\frac{\omega_-\omega_+}{mB}\right)=0, \tag{37}$$

and

$$\frac{(\omega_-+mB)(\omega_+-mB)}{mB}=m(A-B)+\omega_{in}. \tag{38}$$

Then

$$i\dot{f}_{in}^{(+)} + maf_{in}^{(+)} = (M_{in} + \omega_{in})\exp\left[-i\omega_+\eta - \left(\frac{i\omega_-}{\rho}\ln[2\cosh\rho\eta]\right)\right] \times F(a+1, b-1, c, z). \tag{39}$$

For positive frequency *out* mode

$$\begin{aligned}
&\left[i\frac{d}{d\eta} + m(A + B\tanh(\rho\eta))\right] f_{out}^{(+)}(\eta, a, b, c) \\
&\quad = \exp\left[-i\omega_+\eta - \left(\frac{i\omega_-}{\rho}\ln[2\cosh\rho\eta]\right)\right] \\
&\qquad \times \left[\left((\omega_+ + 2z\omega_- - \omega_-) + (\omega_- + mB)\left[-\frac{\omega_+ - mB}{mB} - 2z\right]\right.\right. \\
&\qquad \left. + m(A-B) + 2mBz\right) F(a, b, a+b-c+1, 1-z) \\
&\qquad \left. + \frac{(\omega_- + mB)(\omega_+ + mB)}{mB} F(a+1, b-1, a+b-c+1, 1-z)\right] \\
&\quad = \exp\left[-i\omega_+\eta - \left(\frac{i\omega_-}{\rho}\ln[2\cosh\rho\eta]\right)\right] \\
&\qquad \times \left[\left(mA - \frac{\omega_-\omega_+}{mB}\right) F(a, b, a+b-c+1, 1-z)\right. \\
&\qquad \left. + \frac{(\omega_- + mB)(\omega_+ + mB)}{mB} F(a+1, b-1, a+b-c+1, 1-z)\right].
\end{aligned}$$

Then

$$i\dot{f}_{out}^{(+)} + maf_{out}^{(+)} = (M_{out} + \omega_{out})\exp\left[-i\omega_+\eta - \left(\frac{i\omega_-}{\rho}\ln[2\cosh\rho\eta]\right)\right] \times F(a+1, b-1, a+b-c+1, 1-z). \tag{40}$$

Similarly for negative frequency modes we have

$$-i\dot{f}_{in}^{(-)} + maf_{in}^{(-)} = (M_{in} + \omega_{in})\exp\left[-i\omega_-\eta - \left(\frac{i\omega_+}{\rho}\ln[2\cosh\rho\eta]\right)\right] \times F(a-c+2, b-c, 2-c, z). \tag{41}$$

$$-i\dot{f}_{out}^{(-)} + maf_{out}^{(-)} = (M_{out} + \omega_{out})\exp\left[i\omega_-\eta + \left(\frac{i\omega_+}{\rho}\ln[2\cosh\rho\eta]\right)\right] \times (c-b+1, c-a-1, 1+c-a-b, 1-z) \tag{42}$$

then the four component spinors (24)–(27) take the forms

$$U^{in}(k,1,\mathbf{x},\eta)=N_{in}\exp\left[i\mathbf{k}\cdot\mathbf{x}-i\omega_+\eta-\left(\frac{i\omega_-}{\rho}\ln[2\cosh\rho\eta]\right)\right]$$
$$\times\begin{pmatrix}\begin{pmatrix}M_{in}+\omega_{in}\\0\end{pmatrix}F(a+1,b-1,c,z)\\ \begin{pmatrix}k_3\\k_1+ik_2\end{pmatrix}F(a,b,c,z)\end{pmatrix},\tag{43}$$

$$U^{in}(k,2,\mathbf{x},\eta)=N_{in}\exp\left[i\mathbf{k}\cdot\mathbf{x}-i\omega_+\eta-\left(\frac{i\omega_-}{\rho}\ln[2\cosh\rho\eta]\right)\right]$$
$$\times\begin{pmatrix}\begin{pmatrix}0\\M_{in}+\omega_{in}\end{pmatrix}F(a+1,b-1,c,z)\\ \begin{pmatrix}k_1-ik_2\\-k_3\end{pmatrix}F(a,b,c,z)\end{pmatrix},\tag{44}$$

$$V^{in}(k,1,\mathbf{x},\eta)=N_{in}\exp\left[-i\mathbf{k}\cdot\mathbf{x}-i\omega_-\eta-\left(\frac{i\omega_+}{\rho}\ln[2\cosh\rho\eta]\right)\right]$$
$$\times\begin{pmatrix}\begin{pmatrix}k_3\\k_1+ik_2\end{pmatrix}F(a-c+1,b-c+1,2-c,z)\\ \begin{pmatrix}M_{in}+\omega_{in}\\0\end{pmatrix}F(a-c+2,b-c,2-c,z)\end{pmatrix},\tag{45}$$

$$V^{in}(k,2,\mathbf{x},\eta)=N_{in}\exp\left[-i\mathbf{k}\cdot\mathbf{x}-i\omega_-\eta-\left(\frac{i\omega_+}{\rho}\ln[2\cosh\rho\eta]\right)\right]$$
$$\times\begin{pmatrix}\begin{pmatrix}k_1-ik_2\\-k_3\end{pmatrix}F(a-c+1,b-c+1,2-c,z)\\ \begin{pmatrix}0\\M_{in}+\omega_{in}\end{pmatrix}F(a-c+2,b-c,2-c,z)\end{pmatrix},\tag{46}$$

where $k_\pm=k_1\pm ik_2$. Another set of spinors which behave as Minkowskian spinors in the remote future ($\eta\to+\infty$) are

$$U^{out}(k,1,\mathbf{x},\eta)$$
$$=N_{out}\exp\left[i\mathbf{k}\cdot\mathbf{x}-i\omega_+\eta-\left(\frac{i\omega_-}{\rho}\ln[2\cosh\rho\eta]\right)\right]$$
$$\times\begin{pmatrix}\begin{pmatrix}M_{out}+\omega_{out}\\0\end{pmatrix}F(a+1,b-1,a+b-c+1,1-z)\\ \begin{pmatrix}k_3\\k_1+ik_2\end{pmatrix}F(a,b,a+b-c+1,1-z)\end{pmatrix},\tag{47}$$

$$
\begin{aligned}
&U^{out}(k, 2, \mathbf{x}, \eta) \\
&\quad = N_{out} \exp\left[i\mathbf{k}\cdot\mathbf{x} - i\omega_+\eta - \left(\frac{i\omega_-}{\rho}\ln[2\cosh\rho\eta]\right)\right] \\
&\qquad \times \begin{pmatrix} \begin{pmatrix} 0 \\ M_{out}+\omega_{out} \end{pmatrix} F(a+1, b-1, a+b-c+1, 1-z) \\ \begin{pmatrix} k_1 - ik_2 \\ -k_3 \end{pmatrix} F(a, b, a+b-c+1, 1-z) \end{pmatrix},
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
&V^{out}(k, 1, \mathbf{x}, \eta) \\
&\quad = N_{out} \exp\left[-i\mathbf{k}\cdot\mathbf{x} - i\omega_-\eta + \left(\frac{i\omega_+}{\rho}\ln[2\cosh\rho\eta]\right)\right] \\
&\qquad \times \begin{pmatrix} \begin{pmatrix} k_3 \\ k_1 + ik_2 \end{pmatrix} F(c-a, c-b, c-a-b+1, 1-z) \\ \begin{pmatrix} M_{out}+\omega_{out} \\ 0 \end{pmatrix} F(c-a+1, c-b-1, c-a-b+1, 1-z) \end{pmatrix},
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
&V^{out}(k, 2, \mathbf{x}, \eta) \\
&\quad = N_{out} \exp\left[-i\mathbf{k}\cdot\mathbf{x} - i\omega_-\eta + \left(\frac{i\omega_+}{\rho}\ln[2\cosh\rho\eta]\right)\right] \\
&\qquad \times \begin{pmatrix} \begin{pmatrix} k_1 - ik_2 \\ -k_3 \end{pmatrix} F(c-a, c-b, c-a-b+1, 1-z) \\ \begin{pmatrix} 0 \\ M_{out}+\omega_{out} \end{pmatrix} F(c-a+1, c-b-1, c-a-b+1, 1-z) \end{pmatrix}.
\end{aligned} \tag{50}
$$

In the limit $\eta \to -\infty$

$$
U^{in}(k, 1, \mathbf{x}, \eta) \approx N_{in} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_{in}\eta} \begin{pmatrix} M_{in}+\omega_{in} \\ 0 \\ k_3 \\ k_1 + ik_2 \end{pmatrix}, \tag{51}
$$

$$
U^{in}(k, 2, \mathbf{x}, \eta) \approx N_{in} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_{in}\eta} \begin{pmatrix} 0 \\ M_{in}+\omega_{in} \\ k_1 - ik_2 \\ -k_3 \end{pmatrix}, \tag{52}
$$

$$
V^{in}(k, 1, \mathbf{x}, \eta) \approx N_{in} e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega_{in}\eta} \begin{pmatrix} k_3 \\ k_1 + ik_2 \\ M_{in}+\omega_{in} \\ 0 \end{pmatrix}, \tag{53}
$$

$$V^{in}(k,2,\mathbf{x},\eta) \approx N_{in} e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega_{in}\eta} \begin{pmatrix} k_1 - ik_2 \\ -k_3 \\ 0 \\ M_{in}+\omega_{in} \end{pmatrix}. \tag{54}$$

Also as $\eta \to +\infty$

$$U^{out}(k,1,\mathbf{x},\eta) \approx N_{out} e^{i\mathbf{k}\cdot\mathbf{x}-i\omega_{out}\eta} \begin{pmatrix} M_{out}+\omega_{out} \\ 0 \\ k_3 \\ k_1 + ik_2 \end{pmatrix}, \tag{55}$$

$$U^{out}(k,2,\mathbf{x},\eta) \approx N_{out} e^{i\mathbf{k}\cdot\mathbf{x}-i\omega_{out}\eta} \begin{pmatrix} 0 \\ M_{out}+\omega_{out} \\ k_1 - ik_2 \\ -k_3 \end{pmatrix}, \tag{56}$$

$$V^{out}(k,1,\mathbf{x},\eta) \approx N_{out} e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega_{out}\eta} \begin{pmatrix} k_3 \\ k_1 - ik_2 \\ M_{out}+\omega_{out} \\ 0 \end{pmatrix}, \tag{57}$$

$$V^{out}(k,2,\mathbf{x},\eta) \approx N_{out} e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega_{out}\eta} \begin{pmatrix} k_1 - ik_2 \\ -k_3 \\ 0 \\ M_{out}+\omega_{out} \end{pmatrix}. \tag{58}$$

3 Particle Creation Rate

Let assume that $\{U_{in}(\mathbf{k},d,\eta), V_{in}(\mathbf{k},d,\eta)\}$ and $\{U_{out}(\mathbf{k},d,\eta), V_{out}(\mathbf{k},d,\eta)\}$ are two complete set of mode solutions of Dirac equation which define particles and antiparticles in asymptotic regions. The Dirac field operator can be written as

$$\psi(x) = \int d^3\mathbf{k} \sum_d \big[a_{in}(\mathbf{k},d) U_{in}(\mathbf{k},d,\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + b^{\dagger}_{in}(-\mathbf{k},d) V_{in}(\mathbf{k},d,\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}\big], \tag{59}$$

and

$$\psi(x) = \int d^3\mathbf{k} \sum_d \big[a_{out}(\mathbf{k},d) U_{out}(\mathbf{k},d,\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + b^{\dagger}_{out}(-\mathbf{k},d) V_{out}(\mathbf{k},d,\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}\big], \tag{60}$$

here a_{in}, b_{in} and a_{out}, b_{out} are respectively annihilation of particles and antiparticles. Since each set is complete we can write down one set in terms of another

$$U_{in}(\mathbf{k},d,\eta) = \sum_{d'} \alpha_{dd'} U_{out}(\mathbf{k},d',\eta) + \beta_{dd'} V_{out}(\mathbf{k},d',\eta), \tag{61}$$

$$V_{in}(\mathbf{k},d,\eta) = \sum_{d'} \varrho_{dd'} U_{out}(\mathbf{k},d',\eta) + \sigma_{dd'} V_{out}(\mathbf{k},d',\eta). \tag{62}$$

Then the in and out creation operator of particles and antiparticles are related by

$$a_{out}(\mathbf{k},d)=\sum_{d'}\alpha_{dd'}a_{in}(\mathbf{k},d')+\varrho_{dd'}b^{\dagger}_{in}(-\mathbf{k},d'),\tag{63}$$

$$b_{out}(\mathbf{k},d)=\sum_{d'}\sigma^{*}_{dd'}b_{in}(\mathbf{k},d')+\beta^{*}_{dd'}a^{\dagger}_{in}(-\mathbf{k},d')\tag{64}$$

orthogonality of spinors yields

$$\sum_{d'}|\alpha_{dd'}|^2+|\beta_{dd'}|^2=1,\tag{65}$$

$$\sum_{d'}|\varrho_{dd'}|^2+|\sigma_{dd'}|^2=1.\tag{66}$$

The expectation value of the out particles in the in vacuum (i.e., created particles) is

$$N^{p}=\langle 0_{in}|a^{\dagger}_{out}(\mathbf{k},d)a_{out}(\mathbf{k},d)|0_{in}\rangle=\sum_{d'}|\rho_{dd'}|^2.\tag{67}$$

Similarly the expectation value of out antiparticles in the in vacuum (i.e., created antiparticles) is

$$N^{a}=\langle 0_{in}|b^{\dagger}_{out}(\mathbf{k},d)b_{out}(\mathbf{k},d)|0_{in}\rangle=\sum_{d'}|\beta_{dd'}|^2.\tag{68}$$

Then total number of particles created is

$$N=\sum_{d'}|\rho_{dd'}|^2+\sum_{d'}|\beta_{dd'}|^2.\tag{69}$$

Now we calculate the particle creation rate by the expanding universe. Using the linear transformation properties of hypergeometric functions [10]

$$\begin{aligned}F(a,b,c,z)&=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1,1-z)\\&\quad+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\\&\quad\times F(c-a,c-b,c-a-b+1,1-z),\end{aligned}\tag{70}$$

and

$$F(a,b,c;z)=(1-z)^{c-a-b}F(c-a,c-b,c;z),\tag{71}$$

we can evaluate Bogolubov transformation between $f_{in}(\eta)$ and $f_{out}(\eta)$. At $\eta\to+\infty$ we get

$$\begin{aligned}f^{(+)}_{in}(\eta)&\approx\frac{\Gamma(1-\frac{i\omega_{in}}{\rho})\Gamma(-\frac{i\omega_{out}}{\rho})}{\Gamma(1-\frac{i\omega_{+}}{\rho}-\frac{imB}{\rho})\Gamma(-\frac{i\omega_{+}}{\rho}+\frac{imB}{\rho})}e^{-i\omega_{out}\eta}\\&\quad+\frac{\Gamma(1-\frac{i\omega_{in}}{\rho})\Gamma(\frac{i\omega_{out}}{\rho})}{\Gamma(1+\frac{i\omega_{-}}{\rho}-\frac{imB}{\rho})\Gamma(\frac{i\omega_{-}}{\rho}+\frac{imB}{\rho})}e^{i\omega_{out}\eta},\end{aligned}\tag{72}$$

and

$$f_{in}^{(-)}(\eta) \approx \frac{\Gamma(1+\frac{i\omega_{in}}{\rho})\Gamma(\frac{i\omega_{out}}{\rho})}{\Gamma(1-\frac{i\omega_{-}}{\rho}+\frac{imB}{\rho})\Gamma(-\frac{i\omega_{-}}{\rho}-\frac{imB}{\rho})} e^{-i\omega_{out}\eta} + \frac{\Gamma(1+\frac{i\omega_{in}}{\rho})\Gamma(-\frac{i\omega_{out}}{\rho})}{\Gamma(1+\frac{i\omega_{+}}{\rho}+\frac{imB}{\rho})\Gamma(\frac{i\omega_{+}}{\rho}-\frac{imB}{\rho})} e^{i\omega_{out}\eta}. \tag{73}$$

Then the rates of created particles and antiparticles are the same. Using the orthogonality of the solutions we obtain the creation rates

$$N_p = N_a$$

$$= \left(\left|\frac{\Gamma(1-\frac{i\omega_{-}}{\rho}+\frac{imB}{\rho})\Gamma(-\frac{i\omega_{-}}{\rho}-\frac{imB}{\rho})}{\Gamma(1+\frac{i\omega_{+}}{\rho}+\frac{imB}{\rho})\Gamma(\frac{i\omega_{+}}{\rho}-\frac{imB}{\rho})}\right|^2+1\right)^{-1} \tag{74}$$

$$= \left(\frac{(\omega_{-}-mB)(\omega_{+}-mB)}{(\omega_{-}+mB)(\omega_{+}+mB)}\frac{\sinh\frac{\pi}{\rho}(\omega_{+}+mB)\sinh\frac{\pi}{\rho}(\omega_{+}-mB)}{\sinh\frac{\pi}{\rho}(\omega_{-}+mB)\sinh\frac{\pi}{\rho}(\omega_{-}-mB)}+1\right)^{-1}. \tag{75}$$

$$= \frac{\Omega_{+}\sinh\frac{\pi}{\rho}(\omega_{-}+mB)\sinh\frac{\pi}{\rho}(\omega_{-}-mB)}{\Omega_{-}\sinh\frac{\pi}{\rho}(\omega_{+}+mB)\sinh\frac{\pi}{\rho}(\omega_{+}-mB)+\Omega_{+}\sinh\frac{\pi}{\rho}(\omega_{-}+mB)\sinh\frac{\pi}{\rho}(\omega_{-}-mB)}, \tag{76}$$

where

$$\Omega_{\pm} = m(A+B) \pm \omega_{out}. \tag{77}$$

In the massless case limit no particle production occurs. In the weak expansion limit, i.e., $\rho \to 0$

$$N_p = N_a \approx e^{-\frac{2\pi}{\rho}\omega_{in}}, \tag{78}$$

which is a thermal distribution.

Here we compare Dirac case with scalar field. For scale factor (28) particle creation rate for scalar particles is [11]

$$n_k = \frac{\cosh\pi(\frac{2\bar{\Omega}}{\rho})+\cosh\pi(\frac{2\Omega_{-}}{\rho})}{\cosh\pi(\frac{2\Omega_{+}}{\rho})-\cosh\pi(\frac{2\Omega_{-}}{\rho})} \tag{79}$$

where

$$\bar{\Omega} = [m^2B^2-\rho^2/4]^{1/2}. \tag{80}$$

Similar to Dirac case we have the thermal distribution in the weak expansion limit

$$n_k \approx e^{-2\frac{\pi}{\rho}(\Omega_{+}-\bar{\Omega})}. \tag{81}$$

The creation of high mass particles for $m \gg \rho$, approaches zero sharply, which is reasonable, because production of high energy particles needs much more changing of gravitational field.

Appendix: Feynman Propagator

The Dirac field operator can be written as

$$\begin{aligned}\psi(\mathbf{x},\eta) &= \psi^{(+)}(\mathbf{x},\eta)+\psi^{(-)}(\mathbf{x},\eta)\\ &= \int d^3\mathbf{k}\sum_d [a(\mathbf{k},d)U(\mathbf{k},d,\eta)e^{i\mathbf{k}\cdot\mathbf{x}}+b^{\dagger}(-\mathbf{k},d)V(\mathbf{k},d,\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}],\end{aligned} \tag{82}$$

where the orthonormalization relations between the spinors are

$$\langle U(\mathbf{k}',d',\eta),U(\mathbf{k},d,\eta)\rangle = \langle V(\mathbf{k}',d',\eta),V(\mathbf{k},d,\eta)\rangle = \delta_{dd'}\delta^3(\mathbf{k}'-\mathbf{k}), \tag{83}$$

$$\langle U(\mathbf{k}',d',\eta),V(\mathbf{k},d,\eta)\rangle = \langle V(\mathbf{k}',d',\eta),U(\mathbf{k},d,\eta)\rangle = 0. \tag{84}$$

Standard definition of the Feynman propagator is

$$\begin{aligned}S_F(\eta,\eta',\mathbf{x}-\mathbf{x}') &= i\langle 0|T(\psi(x),\overline{\psi}(x)|0\rangle\\ &= \theta(\eta-\eta')S^{(+)}(\eta,\eta',\mathbf{x}-\mathbf{x}')-\theta(\eta'-\eta)S^{(-)}(\eta',\eta,\mathbf{x}-\mathbf{x}')\end{aligned} \tag{85}$$

where $S^{(\pm)}$ are partial anticommutator functions [12]

$$S^{(\pm)}(\eta,\eta',\mathbf{x}-\mathbf{x}') = i\{\psi^{(\pm)}(\mathbf{x},\eta),\overline{\psi}^{(\pm)}(\mathbf{x}',\eta')\}. \tag{86}$$

calculation of Feynman propagator generally is rather complicated. In some models of spacetimes because of simplicity of the solutions, for example de Sitter, Green functions can be calculated straightforwardly [12, 13].

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